Part 6: Structured Prediction and Energy Minimization (1/2)

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Providence, 21st June 2012

Prediction Problem

$$y^* = f(x) = \operatorname*{argmax}_{y \in \mathcal{Y}} g(x, y)$$

- $g(x, y) = p(y|x)$, factor graphs/MRF/CRF,
- $g(x, y) = -E(y; x, w)$, factor graphs/MRF/CRF,
- $g(x, y) = \langle w, \psi(x, y) \rangle$, linear model (e.g. multiclass SVM),

$\rightarrow$ difficulty: $\mathcal{Y}$ finite but very large

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Prediction Prblem (cont)

Definition (Optimization Problem)

Given $(g, \mathcal{Y}, \mathcal{G}, x)$, with *feasible set* $\mathcal{Y} \subseteq \mathcal{G}$ over *decision domain* $\mathcal{G}$, and given an input instance $x \in \mathcal{X}$ and an *objective function* $g : \mathcal{X} \times \mathcal{G} \to \mathbb{R}$, find the optimal value

$$\alpha = \sup_{y \in \mathcal{Y}} g(x, y),$$

and, if the supremum exists, find an *optimal solution* $y^* \in \mathcal{Y}$ such that $g(x, y^*) = \alpha$.

The feasible set

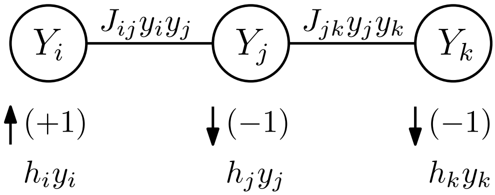
Ingredients

- Decision domain $\mathcal{G}$,
  typically simple ($\mathcal{G} = \mathbb{R}^d$, $\mathcal{G} = 2^V$, etc.)
- Feasible set $\mathcal{Y} \subseteq \mathcal{G}$,
  defining the problem-specific structure
- Objective function $g : \mathcal{X} \times \mathcal{G} \to \mathbb{R}$.

Terminology

- $\mathcal{Y} = \mathcal{G}$: *unconstrained* optimization problem,
- $\mathcal{G}$ finite: *discrete* optimization problem,
- $\mathcal{G} = 2^\Sigma$ for ground set Σ: *combinatorial* optimization problem,
- $\mathcal{Y} = \emptyset$: *infeasible* problem.

Example: Feasible Sets (cont)



- *Ising model* with external field
- Graph $G = (V, E)$
- "External field": $h \in \mathbb{R}^V$
- Interaction matrix: $J \in \mathbb{R}^{V \times V}$
- Objective, defined on $y_i \in \{-1, 1\}$

$$g(y) = h_i y_i + h_j y_j + h_k y_k + \frac{1}{2} J_{ij} y_i y_j + \frac{1}{2} J_{jk} y_j y_k$$

Example: Feasible Sets (cont)

Ising model with external field

$$\mathcal{Y} = \mathcal{G} = \{-1, +1\}^V$$
$$g(y) = \frac{1}{2} \sum_{(i,j) \in E} J_{i,j} y_i y_j + \sum_{i \in V} h_i y_i$$

- Unconstrained
- Objective function contains quadratic terms

Example: Feasible Sets (cont)

$$
\begin{aligned}
\mathcal{G} &= \{0,1\}^{(V\times\{-1,+1\})\cup(E\times\{-1,+1\}\times\{-1,+1\})},\\
\mathcal{Y} &= \{y\in\mathcal{G} : \forall i\in V : y_{i,-1}+y_{i,+1}=1,\\
&\qquad \forall (i,j)\in E : y_{i,j,+1,+1}+y_{i,j,+1,-1} = y_{i,+1},\\
&\qquad \forall (i,j)\in E : y_{i,j,-1,+1}+y_{i,j,-1,-1} = y_{i,-1}\},\\
g(y) &= \frac{1}{2}\sum_{(i,j)\in E} J_{i,j}(y_{i,j,+1,+1}+y_{i,j,-1,-1})\\
&\qquad -\frac{1}{2}\sum_{(i,j)\in E} J_{i,j}(y_{i,j,+1,-1}+y_{i,j,-1,+1})\\
&\qquad +\sum_{i\in V} h_i(y_{i,+1}-y_{i,-1})
\end{aligned}
$$

- Constrained, more variables
- Objective function contains linear terms only

Evaluating f: what do we want?

$$f(x) = \operatorname*{argmax}_{y \in \mathcal{Y}} g(x, y)$$

For evaluating $f(x)$ we want an algorithm that

1. is *general*: applicable to all instances of the problem,
2. is *optimal*: provides an optimal y^*,
3. has good *worst-case complexity*: for all instances the runtime and space is acceptably bounded,
4. is *integral*: its solutions are restricted to $\mathcal{Y}$,
5. is *deterministic*: its results and runtime are reproducible and depend on the input data only.

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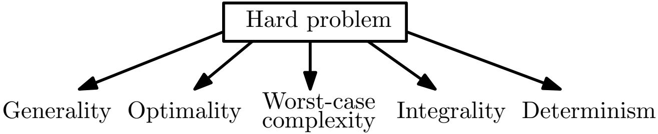
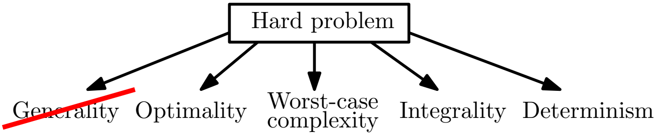
wanting all of them $\rightarrow$ impossible

Giving up some properties



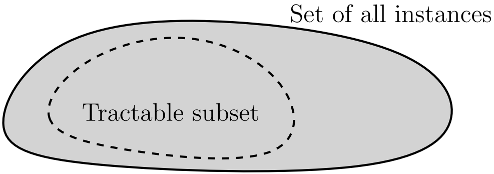
$\rightarrow$ giving up one or more properties

- allows us to design algorithms satisfying the remaining properties
- might be sufficient for the task at hand


Hard problem
Generality
Optimality
Worst-case complexity
Integrality
Determinism

Giving up Generality

- Identify an interesting and tractable subset of instances



Example: MAP Inference in Markov Random Fields

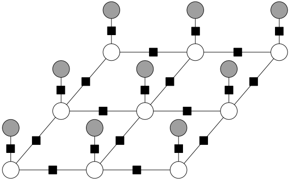
Although NP-hard in general, it is tractable...

- with low tree-width (Lauritzen, Spiegelhalter, 1988)
- with binary states, pairwise submodular interactions (Boykov, Jolly, 2001)
- with binary states, pairwise interactions (only), planar graph structure (Globerson, Jaakkola, 2006)
- with submodular pairwise interactions (Schlesinger, 2006)
- with $\mathcal{P}^n$-Potts higher order factors (Kohli, Kumar, Torr, 2007)
- with perfect graph structure (Jebara, 2009)

Binary Graph-Cuts

- Energy function: unary and pairwise

$$E(y;x,w) = \sum_{F \in \mathcal{F}_1} E_F(y_F;x,w_{t_F}) + \sum_{F \in \mathcal{F}_2} E_F(y_F;x,w_{t_F})$$



- Restriction 1 (wlog)

$$E_F(y_i;x,w_{t_F}) \geq 0$$

- Restriction 2 (regular/submodular/attractive)

$$\begin{aligned} E_F(y_i,y_j;x,w_{t_F}) &= 0, && \text{if } y_i = y_j, \\ E_F(y_i,y_j;x,w_{t_F}) &= E_F(y_j,y_i;x,w_{t_F}) \geq 0, && \text{otherwise.} \end{aligned}$$

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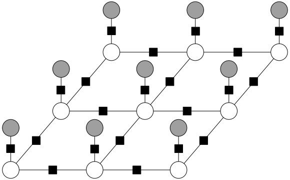

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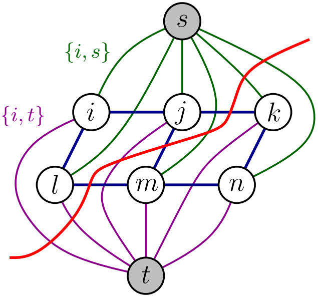
Binary Graph-Cuts (cont)

- Construct auxiliary undirected graph
- One node $\{i\}_{i \in V}$ per variable
- Two extra nodes: source s, sink t
- Edges

Edge	Graph cut weight
$\{i,j\}$	$E_F(y_i = 0, y_j = 1; x, w_{t_F})$
$\{i,s\}$	$E_F(y_i = 1; x, w_{t_F})$
$\{i,t\}$	$E_F(y_i = 0; x, w_{t_F})$

- Find linear s-t-mincut
- Solution defines optimal binary labeling of the original energy minimization problem



Example: Figure-Ground Segmentation

Input image (`http://pdphoto.org`)

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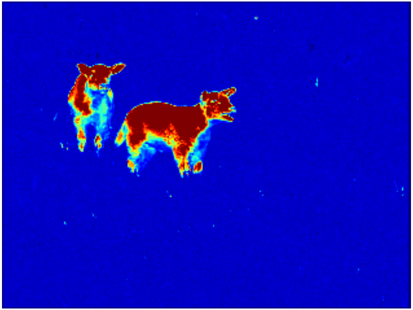

Color model log-odds

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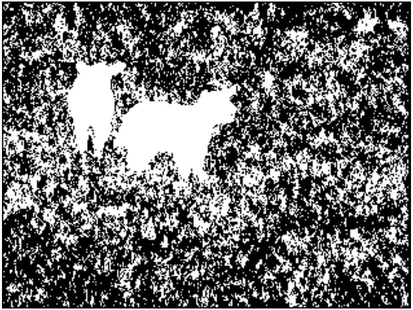
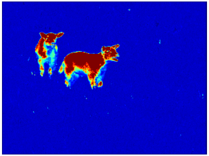


Independent decisions

Example: Figure-Ground Segmentation



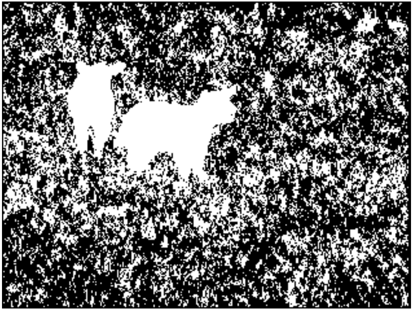
$$g(x, y, w) = \sum_{i \in V} \log p(y_i | x_i) + w \sum_{(i,j) \in E} C(x_i, x_j) I(y_i \neq y_j)$$

- Gradient strength

$$C(x_i, x_j) = \exp(\gamma \|x_i - x_j\|^2)$$

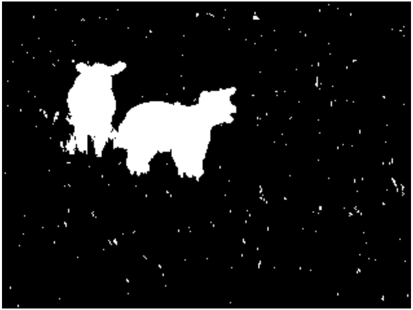
  γ estimated from mean edge strength (Blake et al, 2004)
- $w \geq 0$ controls smoothing

Example: Figure-Ground Segmentation



$w = 0$

Example: Figure-Ground Segmentation



Small $w > 0$

Example: Figure-Ground Segmentation

Medium $w > 0$

Example: Figure-Ground Segmentation

Large $w > 0$

General Binary Case

- Is there a larger class of energies for which binary graph cuts are applicable?
- (Kolmogorov and Zabih, 2004), (Freedman and Drineas, 2005)

Theorem (Regular Binary Energies)

$$E(y;x,w) = \sum_{F\in\mathcal{F}_1} E_F(y_F;x,w_{t_F}) + \sum_{F\in\mathcal{F}_2} E_F(y_F;x,w_{t_F})$$

is a energy function of binary variables containing only unary and pairwise factors. The discrete energy minimization problem $\operatorname{argmin}_y E(y;x,w)$ *is representable as a graph cut problem if and only if all pairwise energy functions* E_F *for* $F\in\mathcal{F}_2$ *with* $F=\{i,j\}$ *satisfy*

$$E_{i,j}(0,0)+E_{i,j}(1,1) \le E_{i,j}(0,1)+E_{i,j}(1,0).$$

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$$E_{i,j}(0,0) + E_{i,j}(1,1) \leq E_{i,j}(0,1) + E_{i,j}(1,0).$$

Example: Class-independent Object Hypotheses

- (Carreira and Sminchisescu, 2010)
  PASCAL VOC 2009/2010 segmentation winner
- Generate class-independent object hypotheses
- Energy (almost) as before

$$g(x, y, w) = \sum_{i \in V} E_i(y_i) + w \sum_{(i,j) \in E} C(x_i, x_j) I(y_i \neq y_j)$$

- Fixed unaries

$$E_i(y_i) = \begin{cases} \infty & \text{if } i \in V_{fg} \text{ and } y_i = 0 \\ \infty & \text{if } i \in V_{bg} \text{ and } y_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

- Test all $w \geq 0$ using *parametric max-flow*
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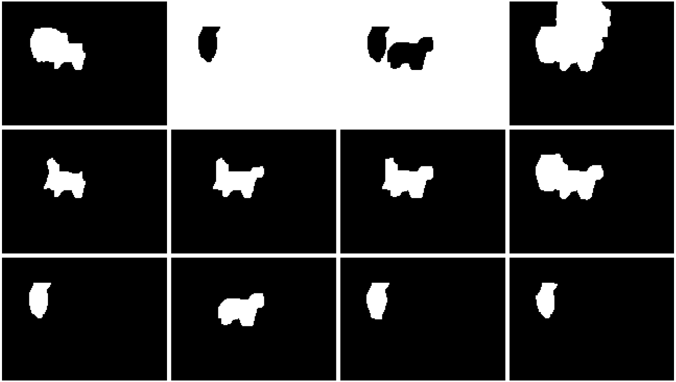
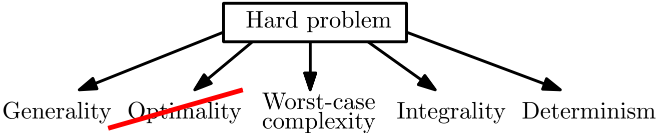


CPMC proposal segmentations (Carreira and Sminchisescu, 2010)


Hard problem
Generality
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Worst-case complexity
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Giving up Optimality

Solving for y^* is hard, but is it necessary?

- pragmatic motivation: in many applications a close-to-optimal solution is good enough
- computational motivation: set of "good" solutions might be large and finding just one element can be easy

For machine learning models

- *modeling error*: we always use the wrong model
- *estimation error*: preference for y^* might be an artifact

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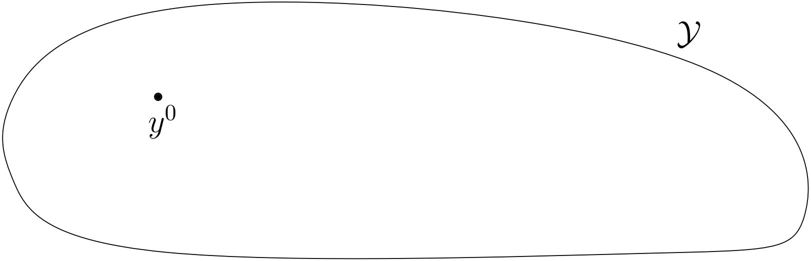
Local Search



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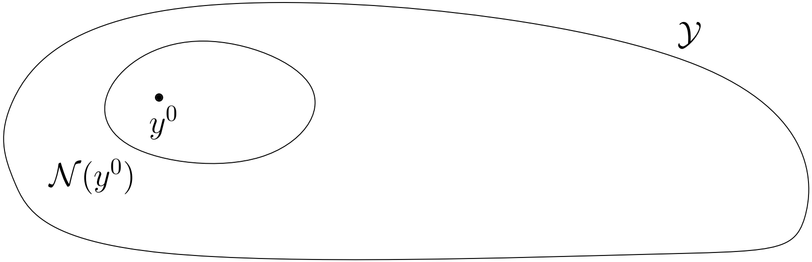

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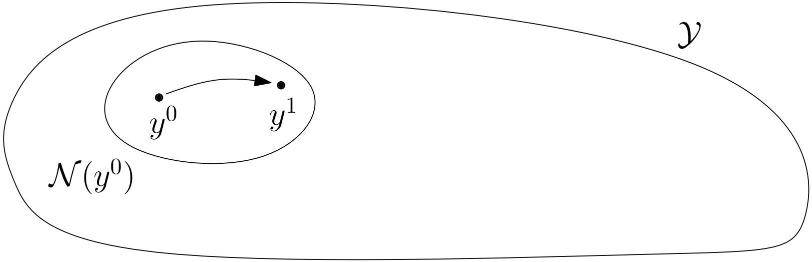

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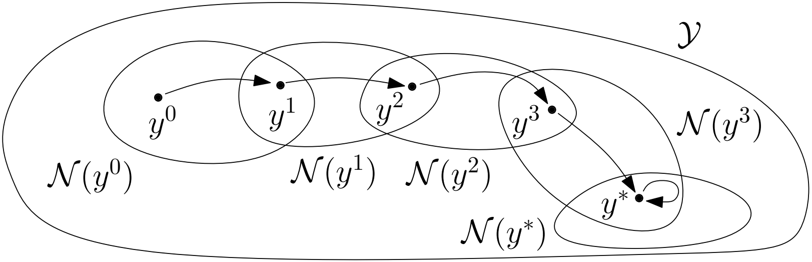

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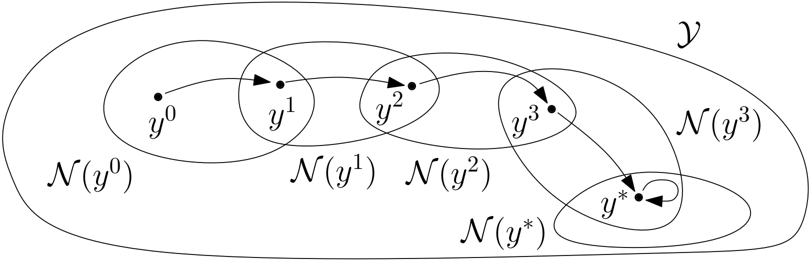
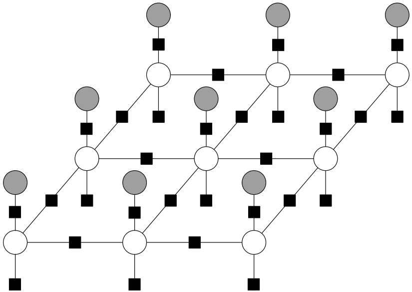


- $\mathcal{N}_t : \mathcal{Y} \to 2^{\mathcal{Y}}$, neighborhood system
- Optimization with respect to $\mathcal{N}_t(y)$ must be tractable:

$$y^{t+1} = \operatorname*{argmax}_{y \in \mathcal{N}_t(y^t)} g(x, y)$$

Example: Iterated Conditional Modes (ICM)

Iterated Conditional Modes (ICM), (Besag, 1986)



- $g(x, y) = \log p(y|x)$
- $y^* = \underset{y \in \mathcal{Y}}{\operatorname{argmax}} \log p(y|x)$
- Neighborhoods $\mathcal{N}_s(y) = \{(y_1, \ldots, y_{s-1}, z_s, y_{s+1}, \ldots, y_S) | z_s \in \mathcal{Y}_s\}$

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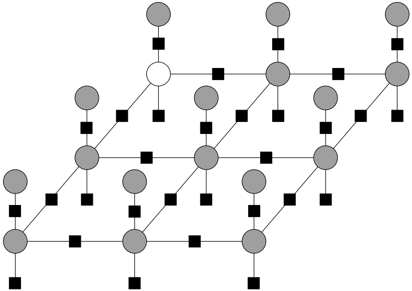

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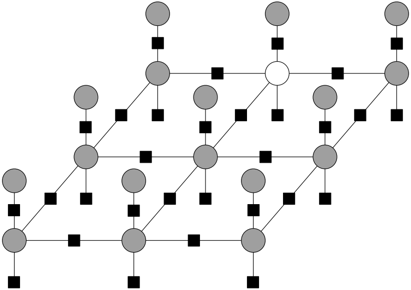


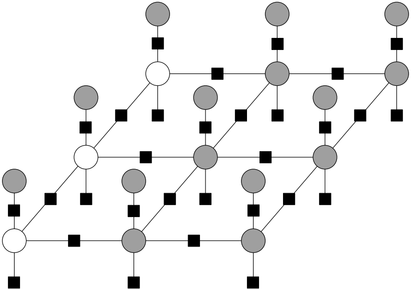
- $y^{t+1} = \operatorname*{argmax}_{y_2 \in \mathcal{Y}_2} \log p(y_1^t, y_2, y_3^t, \dots, y_V^t | x)$

Neighborhood Size

- ICM neighborhood $\mathcal{N}_t(y^t)$: all states reachable from y^t by changing a single variable (Besag, 1986)
- Neighborhood size: in general, larger is better (VLSN, Ahuja, 2000)
- Example: neighborhood along chains

Example: Block ICM

Block Iterated Conditional Modes (ICM)
(Kelm et al., 2006), (Kittler and Föglein, 1984)



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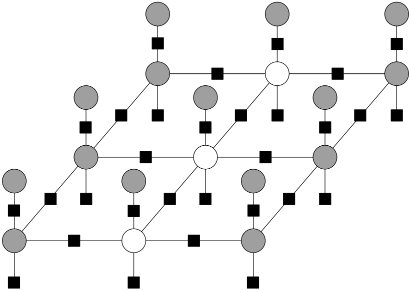

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Example: Multilabel Graph-Cut

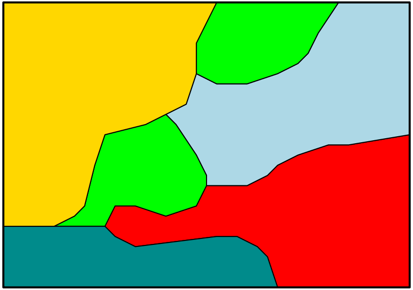
- Binary graph-cuts are not applicable to multilabel energy minimization problems
- (Boykov et al., 2001): two local search algorithms for multilabel problems
- Sequence of binary directed *s*-*t*-mincut problems
- Iteratively improve multilabel solution

α-β Swap Neighborhood

- Select two different labels α and β
- Fix all variables i for which $y_i \notin \{\alpha, \beta\}$
- Optimize over remaining i with $y_i \in \{\alpha, \beta\}$

$$\mathcal{N}_{\alpha,\beta} : \mathcal{Y} \times \mathbb{N} \times \mathbb{N} \rightarrow 2^{\mathcal{Y}},$$
$$\mathcal{N}_{\alpha,\beta}(y, \alpha, \beta) := \{z \in \mathcal{Y} : z_i = y_i \text{ if } y_i \notin \{\alpha, \beta\}, \quad \text{otherwise } z_i \in \{\alpha, \beta\}\}.$$

α-β-swap illustrated



- 5-label problem
- $\alpha - \beta$-swap

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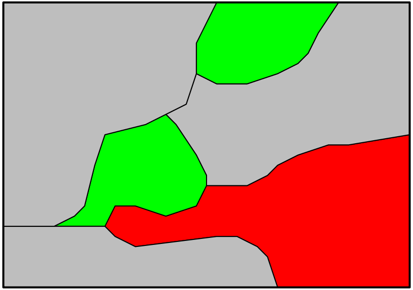

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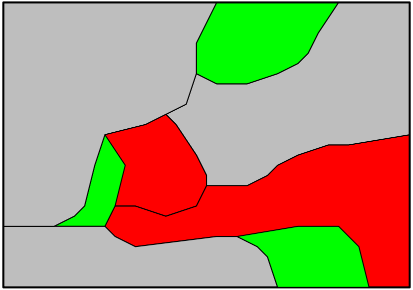

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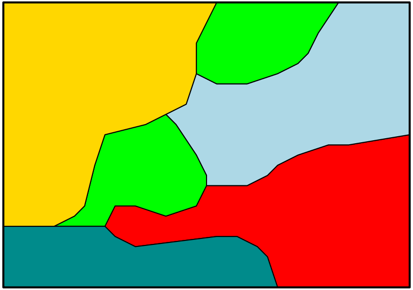
- $\alpha - \beta$-swap

α-β-swap illustrated



- 5-label problem
- $\alpha - \beta$-swap

α-β-swap derivation

$$y^{t+1} = \operatorname*{argmin}_{y \in \mathcal{N}_{\alpha,\beta}(y^t,\alpha,\beta)} E(y;x)$$

- Constant: drop out
- Unary: combine
- Pairwise: binary pairwise

α-β-swap derivation

$$y^{t+1} = \operatorname*{argmin}_{y \in \mathcal{N}_{\alpha,\beta}(y^t,\alpha,\beta)} \sum_{i \in V} E_i(y_i;x) + \sum_{(i,j) \in E} E_{i,j}(y_i,y_j;x)$$

- Constant: drop out
- Unary: combine
- Pairwise: binary pairwise

α-β-swap derivation

$$
\begin{aligned}
y^{t+1} = \operatorname*{argmin}_{y \in \mathcal{N}_{\alpha,\beta}(y^t,\alpha,\beta)} \Bigg[& \sum_{\substack{i \in V, \\ y_i^t \notin \{\alpha,\beta\}}} E_i(y_i^t; x) + \sum_{\substack{i \in V, \\ y_i^t \in \{\alpha,\beta\}}} E_i(y_i; x) \\
& + \sum_{\substack{(i,j) \in E, \\ y_i^t \notin \{\alpha,\beta\}, y_j^t \notin \{\alpha,\beta\}}} E_{i,j}(y_i^t, y_j^t; x) + \sum_{\substack{(i,j) \in E, \\ y_i^t \in \{\alpha,\beta\}, y_j^t \notin \{\alpha,\beta\}}} E_{i,j}(y_i, y_j^t; x) \\
& + \sum_{\substack{(i,j) \in E, \\ y_i^t \notin \{\alpha,\beta\}, y_j^t \in \{\alpha,\beta\}}} E_{i,j}(y_i^t, y_j; x) + \sum_{\substack{(i,j) \in E, \\ y_i^t \in \{\alpha,\beta\}, y_j^t \in \{\alpha,\beta\}}} E_{i,j}(y_i, y_j; x) \Bigg].
\end{aligned}
$$

- Constant: drop out
- Unary: combine
- Pairwise: binary pairwise

α-β-swap derivation

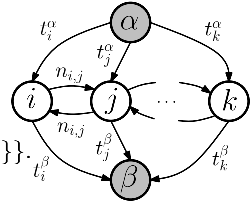
$$
\begin{aligned}
y^{t+1} \quad = \quad \operatorname*{argmin}_{y \in \mathcal{N}_{\alpha,\beta}(y^t,\alpha,\beta)} \Bigg[& \sum_{\substack{i \in V, \\ y_i^t \notin \{\alpha,\beta\}}} E_i(y_i^t; x) + \sum_{\substack{i \in V, \\ y_i^t \in \{\alpha,\beta\}}} E_i(y_i; x) \\
& + \sum_{\substack{(i,j) \in E, \\ y_i^t \notin \{\alpha,\beta\}, y_j^t \notin \{\alpha,\beta\}}} E_{i,j}(y_i^t, y_j^t; x) + \sum_{\substack{(i,j) \in E, \\ y_i^t \in \{\alpha,\beta\}, y_j^t \notin \{\alpha,\beta\}}} E_{i,j}(y_i, y_j^t; x) \\
& + \sum_{\substack{(i,j) \in E, \\ y_i^t \notin \{\alpha,\beta\}, y_j^t \in \{\alpha,\beta\}}} E_{i,j}(y_i^t, y_j; x) + \sum_{\substack{(i,j) \in E, \\ y_i^t \in \{\alpha,\beta\}, y_j^t \in \{\alpha,\beta\}}} E_{i,j}(y_i, y_j; x) \Bigg].
\end{aligned}
$$

- Constant: drop out
- Unary: combine
- Pairwise: binary pairwise

α-β-swap graph construction

- Directed graph $G' = (V', \mathcal{E}')$

$$
\begin{aligned}
V' &= \{\alpha, \beta\} \cup \{i \in V : y_i \in \{\alpha, \beta\}\}, \\
E' &= \{(\alpha, i, t_i^\alpha) : \forall i \in V : y_i \in \{\alpha, \beta\}\} \cup \\
&\quad \{(i, \beta, t_i^\beta) : \forall i \in V : y_i \in \{\alpha, \beta\}\} \cup \\
&\quad \{(i, j, n_{i,j}) : \forall (i,j), (j,i) \in E : y_i, y_j \in \{\alpha, \beta\}\}.
\end{aligned}
$$



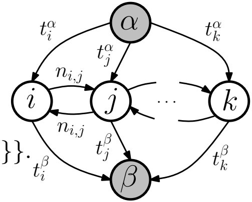
- Edge weights t_i^α, t_i^β, and $n_{i,j}$

$$
\begin{aligned}
n_{i,j} &= E_{i,j}(\alpha, \beta; x) \\
t_i^\alpha &= E_i(\alpha; x) + \sum_{\substack{(i,j) \in \mathcal{E}, \\ y_j \notin \{\alpha,\beta\}}} E_{i,j}(\alpha, y_j; x) \\
t_i^\beta &= E_i(\beta; x) + \sum_{\substack{(i,j) \in \mathcal{E}, \\ y_j \notin \{\alpha,\beta\}}} E_{i,j}(\beta, y_j; x)
\end{aligned}
$$

α-β-swap graph construction

- Directed graph $G' = (V', \mathcal{E}')$

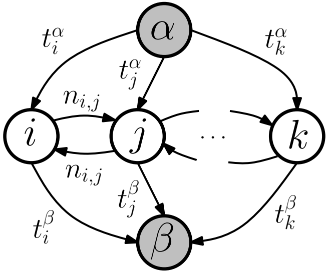
$$\begin{aligned} V' &= \{\alpha, \beta\} \cup \{i \in V : y_i \in \{\alpha, \beta\}\}, \\ E' &= \{(\alpha, i, t_i^\alpha) : \forall i \in V : y_i \in \{\alpha, \beta\}\} \cup \\ &\quad \{(i, \beta, t_i^\beta) : \forall i \in V : y_i \in \{\alpha, \beta\}\} \cup \\ &\quad \{(i, j, n_{i,j}) : \forall (i,j), (j,i) \in E : y_i, y_j \in \{\alpha, \beta\}\}. \end{aligned}$$



- Edge weights t_i^α, t_i^β, and $n_{i,j}$

$$\begin{aligned} n_{i,j} &= E_{i,j}(\alpha, \beta; x) \\ t_i^\alpha &= E_i(\alpha; x) + \sum_{\substack{(i,j) \in \mathcal{E}, \\ y_j \notin \{\alpha, \beta\}}} E_{i,j}(\alpha, y_j; x) \\ t_i^\beta &= E_i(\beta; x) + \sum_{\substack{(i,j) \in \mathcal{E}, \\ y_j \notin \{\alpha, \beta\}}} E_{i,j}(\beta, y_j; x) \end{aligned}$$

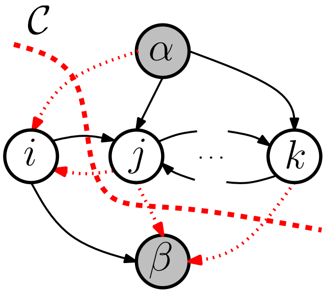
α-β-swap move



- Side of cut determines $y_i \in \{\alpha, \beta\}$
- Iterate all possible (α, β) combinations
- Semi-metric requirement on pairwise energies

$$E_{i,j}(y_i, y_j; x) = 0 \Leftrightarrow y_i = y_j$$
$$E_{i,j}(y_i, y_j; x) = E_{i,j}(y_j, y_i; x) \geq 0$$

α-β-swap move



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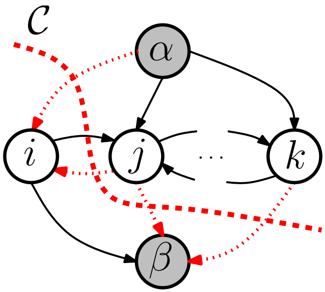

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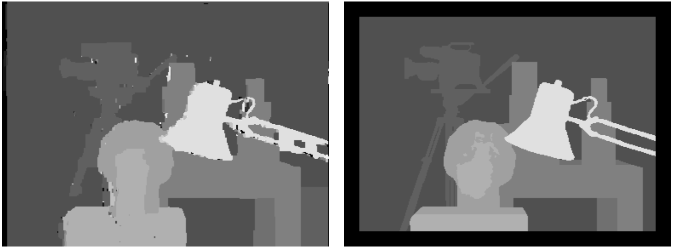
Example: Stereo Disparity Estimation

- Infer depth from two images
- Discretized multi-label problem
- α-expansion solution close to optimal

Example: Stereo Disparity Estimation



- Infer depth from two images
- Discretized multi-label problem
- α-expansion solution close to optimal

Model Reduction

- Energy minimization problem: many decision to make jointly
- Model reduction
  1. Fix a subset of decisions
  2. Optimize the smaller remaining model
- Example: forcing $y_i = y_j$ for pairs (i, j)

Example: Superpixels in Labeling Problems

Input image: 500-by-375 pixels (187,500 decisions)

Example: Superpixels in Labeling Problems

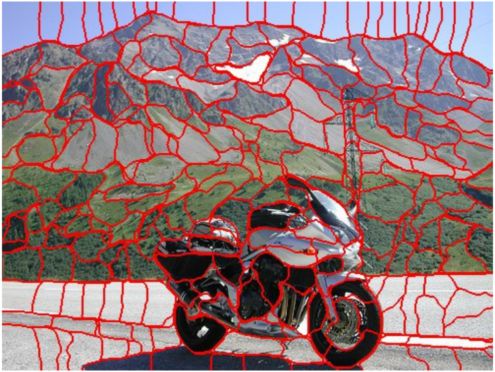


Image with 149 superpixels (149 decisions)