Christian Hoffmann (snick@cs.tu-berlin.de)
Marco Kunze (makunze@cs.tu-berlin.de)
Sebastian Nowozin (nowozin@cs.tu-berlin.de)

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Lösungen zum 11. Übungsblatt zur Mafi I

Lösung zu Aufgabe 41:

Die gegebene Gleichung hat 4 Lösungen:

$$
\begin{aligned}
z_0 &= \sqrt[4]{1} \cdot e^{i \cdot \left(\frac{\pi}{8}\right)} = 1 \cdot e^{i \cdot \left(\frac{\pi}{8}\right)} = \cos\left(\frac{\pi}{8}\right) + i \cdot \sin\left(\frac{\pi}{8}\right) \\
z_1 &= \sqrt[4]{1} \cdot e^{i \cdot \left(\frac{\pi}{8} + \frac{\pi}{2}\right)} = 1 \cdot e^{i \cdot \left(\frac{5\pi}{8}\right)} = \cos\left(\frac{5\pi}{8}\right) + i \cdot \sin\left(\frac{5\pi}{8}\right) \\
z_2 &= \sqrt[4]{1} \cdot e^{i \cdot \left(\frac{\pi}{8} + \pi\right)} = 1 \cdot e^{i \cdot \left(\frac{9\pi}{8}\right)} = \cos\left(\frac{9\pi}{8}\right) + i \cdot \sin\left(\frac{9\pi}{8}\right) \\
z_3 &= \sqrt[4]{1} \cdot e^{i \cdot \left(\frac{\pi}{8} + i\frac{3\pi}{2}\right)} = 1 \cdot e^{i \cdot \left(\frac{9\pi}{8}\right)} = \cos\left(\frac{9\pi}{8}\right) + i \cdot \sin\left(\frac{9\pi}{8}\right)
\end{aligned}
$$

Lösung zu Aufgabe 42:

(a) (i)

$$
\begin{aligned}
\sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \\
\Leftrightarrow \frac{e^{x+y} - e^{-x-y}}{2} &= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} \\
\Leftrightarrow e^x e^y - e^{-x} e^{-y} &= \frac{e^x e^y + e^x e^{-y} - e^{-x} e^y - e^{-x} e^{-y} + e^x e^y - e^x e^{-y} + e^{-x} e^y - e^{-x} e^{-y}}{2} \\
\Leftrightarrow e^x e^y - e^{-x} e^{-y} &= \frac{2e^x e^y - 2e^{-x} e^{-y}}{2} \\
\Leftrightarrow e^x e^y - e^{-x} e^{-y} &= e^x e^y - e^{-x} e^{-y}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \\
\Leftrightarrow \frac{e^{x+y} + e^{-x-y}}{2} &= \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} \\
\Leftrightarrow e^x e^y + e^{-x} e^{-y} &= \frac{e^x e^y + e^x e^{-y} + e^{-x} e^y + e^{-x} e^{-y} + e^x e^y - e^x e^{-y} - e^{-x} e^y + e^{-x} e^{-y}}{2} \\
\Leftrightarrow e^x e^y + e^{-x} e^{-y} &= \frac{2e^x e^y + 2e^{-x} e^{-y}}{2} \\
\Leftrightarrow e^x e^y + e^{-x} e^{-y} &= e^x e^y + e^{-x} e^{-y}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \frac{\cosh(\varepsilon) - 1}{\varepsilon} &= \lim_{\varepsilon \to 0} \frac{\frac{e^\varepsilon + e^{-\varepsilon}}{2} - 1}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{e^\varepsilon + e^{-\varepsilon} - 2}{2\varepsilon}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\varepsilon \to 0} \frac{e^{\varepsilon} - e^{-\varepsilon}}{2} \\
&= \frac{1-1}{2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \frac{\sinh(\varepsilon)}{\varepsilon} &= \lim_{\varepsilon \to 0} \frac{\frac{e^{\varepsilon} - e^{-\varepsilon}}{2}}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{e^{\varepsilon} - e^{-\varepsilon}}{2\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{e^{\varepsilon} + e^{-\varepsilon}}{2} \\
&= \frac{1+1}{2} \\
&= 1
\end{aligned}
$$

(c)

$$
\begin{aligned}
(\cosh(x))' &= \lim_{n \to \infty} \frac{\cosh(x+h) - \cosh(x)}{h} \\
&= \lim_{n \to \infty} \frac{\frac{e^{x+h} + e^{-x-h}}{2} - \frac{e^{x} + e^{-x}}{2}}{h} \\
&= \lim_{n \to \infty} \frac{e^{x}e^{h} + e^{-x}e^{-h} - e^{x} + e^{-x}}{2h} \\
&\overset{l'Hospital}{=} \lim_{n \to \infty} \frac{e^{x}e^{h} - e^{-x}e^{-h}}{2} \\
&= \frac{e^{x} - e^{-x}}{2} \\
&= \sinh(x)
\end{aligned}
$$

$$
\begin{aligned}
(\sinh(x))' &= \lim_{n \to \infty} \frac{\sinh(x+h) - \sinh(x)}{h} \\
&= \lim_{n \to \infty} \frac{\frac{e^{x+h} - e^{-x-h}}{2} - \frac{e^{x} - e^{-x}}{2}}{h} \\
&= \lim_{n \to \infty} \frac{e^{x}e^{h} - e^{-x}e^{-h} - e^{x} + e^{-x}}{2h} \\
&\overset{l'Hospital}{=} \lim_{n \to \infty} \frac{e^{x}e^{h} + e^{-x}e^{-h}}{2} \\
&= \frac{e^{x} + e^{-x}}{2} \\
&= \cosh(x)
\end{aligned}
$$

Lösung zu Aufgabe 43:

(a)

$$
f(x) = x^{(x^x)} = e^{\ln x^{(x^x)}} = e^{x^x \ln x} = e^{e^{\ln(x^x)} \ln x} = e^{e^{x \ln x} \ln x}
$$

$$
\begin{aligned}
f'(x) &= e^{e^{x \ln(x)} \ln x} \cdot \left(e^{x \ln(x)} \ln x\right)' \\
&= e^{e^{x \ln(x)} \ln x} \cdot \left(\left(e^{x \ln x}\right)' \cdot \ln(x) + \frac{e^{x \ln(x)}}{x}\right) \\
&= e^{e^{x \ln(x)} \ln x} \cdot \left(e^{x \ln x} \left(\ln(x) + 1\right) \cdot \ln x + \frac{e^{x \ln(x)}}{x}\right)
\end{aligned}
$$

(b)

$$f(x) = arcsinh(x)$$

Uminterpretierung:

$$\begin{aligned} f^{-1}(y) &= arcsinh(y) \\ f(x) &= \sinh(x) \quad \Rightarrow \quad f'(x) = \cosh(x) \end{aligned}$$

$$\begin{aligned} (f^{-1})'(y) &= \frac{1}{f'(f^{-1}(y))} \\ &= \frac{1}{\cosh(f^{-1}(y))} \\ &= \frac{1}{\cosh(arcsinh(y))} \\ &= \frac{1}{\sqrt{1 + (\sinh(arcsinh(y))^2}} \\ &= \frac{1}{\sqrt{1+y^2}} \end{aligned}$$

Rückinterpretierung:

$$f'(x) = \frac{1}{\sqrt{1+x^2}}$$

(c)

$$\begin{aligned} f'(x) &= \frac{\left(x^2 \sin(x)\right)' \left(1+x^4\right) - \left(x^2 \sin(x)\right) \left(1+x^4\right)'}{\left(1+x^4\right)^2} \\ &= \frac{\left(2x \sin(x) + x^2 \cos(x)\right) \left(1+x^4\right) - 4x^5 \sin(x)}{x^8 + 2x^4 + 1} \end{aligned}$$

(d)

$$\begin{aligned} f'(x) &= \left(\frac{x^2-2}{3x}\right)' \cdot e^{\sqrt{x}} + \frac{x^2-2}{3x} \left(e^{\sqrt{x}}\right)' \\ &= \left(\frac{3x^2+6}{9x^2} e^{\sqrt{x}}\right) + \frac{x^2-2}{3x} \cdot \frac{e^{\sqrt{x}}}{2\sqrt{x}} \end{aligned}$$

(e)

$$f'(x) = -\cos\left(\cos(x)\right) \sin(x) \cos\left(\sin(x)\right) - \sin\left(\cos(x)\right) \sin\left(\sin(x)\right) \cos(x)$$

Lösung zu Aufgabe 44:

$$\sum_{n=42}^{\infty} \frac{(3x)^n}{4^n \left(1-\frac{1}{n}\right)^n} = \sum_{n=42}^{\infty} \frac{3^n}{4^n \left(1-\frac{1}{n}\right)^n} \cdot (3x)^n$$

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{4^n \left(1-\frac{1}{n}\right)^n}} &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{4\left(1-\frac{1}{n}\right)}\right)} \\ &= \lim_{n\to\infty} \frac{3}{4\left(1-\frac{1}{n}\right)} \\ &= \lim_{n\to\infty} \frac{3}{4\left(4-\frac{4}{n}\right)} \\ &= \frac{3}{4} \end{aligned}$$

Damit ist $R = \frac{4}{3}$, die Reihe ist für alle $x \in \mathbb{R}$, $x \in (-\frac{4}{3}, \frac{4}{3})$ differenzierbar.
Die Ableitung lautet:

$$f'(x) = \sum_{n=43}^{\infty} n \cdot \left(\frac{3}{4 \cdot \left(1 - \frac{1}{n}\right)} \right)^n \cdot x^{n-1}$$